Randomized Gandy-Păun-Rozenberg machines

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Abstract. An idea of a randomized Gandy–Păun–Rozenberg machine providing a certain abstract implementation of concurrent (parallel) randomized algorithms is introduced. A randomized Gandy–Păun–Rozenberg machine for solving 3-SAT problem in a polynomial time with the low error probability and with subexponential number of indecomposable processors is shown, where this machine assembles a distributed system which then realizes a massively parallel computation.

1 Introduction

We propose and discuss an idea of a randomized Gandy–Păun–Rozenberg machine which is a randomized counterpart of a concept of a Gandy–Păun–Rozenberg machine introduced in [5] and recalled in Section 2 of the present paper, where one finds the connections with membrane computing.

In general randomized Gandy–Păun–Rozenberg machines, briefly called randomized G–P–R machines, are aimed to serve for an abstract implementation of concurrent (parallel) randomized algorithms or to provide some description of these algorithms.

Here one can describe informally a *randomized algorithm* as an algorithm which contains a possibly deterministic algorithm (test) performed for some randomly chosen input data in a possibly polynomial time, where there is known some estimation of error probability of the final result of the test; for a more formal approach see [6]. Randomized algorithms are used when for their tasks there are not known efficient deterministic algorithms.

In particular a goal of randomization of Gandy–Păun–Rozenberg machines, yielding randomized G–P–R machines, is to decrease the exponential expansion of the number of indecomposable processors which appear in computations of G–P–R machines constructed in the manner of [5] to solve NP complete problems in a polynomial time. A decreasing of exponential expansion of the number of indecomposable processors to some subexponential one is achieved with a loss of certainty of a final result which is reached with some error probability in a similar way as in the case of randomized algorithms, where a subexponential time of computations is achieved with a loss of certainty of a final result, cf. [4]. Randomized G–P–R machines are similar to Gandy–Păun–Rozenberg machines except the initial instantaneous descriptions of randomized G–P–R machines contain certain configurations chosen at random. These randomly chosen configurations steer the computations in such a way that subexponential number of indecomposable processors appear in the computations.

The subexponentiality is understood here in the following way. For the length k of input data the time or space complexity measures with respect to k of a given algorithm are called *subexponential* if they are bounded by a function $e^{f(k)}$ for some function $f(k)$ with $\lim_{k\to\infty} \frac{f(k)}{k} = 0$, see [4].

We present in Section 3 an explanatory example of a randomized G–P–R machine which solves 3-SAT problem in a polynomial time with low error probability and with subexponential number of indecomposable processors. The example shows that a parallelized randomization or randomized parallelization of computations is possible, which is explained in the conclusion of the paper.

The example also shows a theoretical (discrete-topological) possibility of a construction of a system, still in Gandy's mechanism frames[1] [2], whose computation process consists of two phases: a phase of assembly of a distributed system (possibly with randomly chosen input data), and then a phase of massively parallel computation realized by this distributed system.

2 Gandy-Păun-Rozenberg machines and their examples

An idea of a Gandy–Păun–Rozenberg machine, briefly G–P–R machine, introduced in [5], is aimed to provide an answer to the question:

what is it an $\mathcal{X}$ possible machine?

for $\mathcal{X}$ ≡ set-theoretically, $\mathcal{X}$ ≡ discrete topologically, and $\mathcal{X}$ ≡ biologically inspired.

The G–P–R machines are the constructs which have common features with or are related to:

— Gandy's machines [2], [9],
— P systems due to Gheorghe Păun (cf. [8]),
— parallel rewriting systems of graphs investigated by Grzegorz Rozenberg himself with scientists cooperating with him, among others, in preparation and editing of many volume *Handbook of graph grammars and computing by graph transformation* [3].

The core of a G–P–R machine is a finite set of rewriting rules for certain finite directed labelled graphs, where these graphs are instantenous descriptions for the computation process realized by the machine.

The conflictless parallel (simultaneous) application of the rewriting rules of a G–P–R machine is realized in Gandy's machine mode (according to Local Causality Principle), where (local) maximality of "causal neighbourhoods" replaces (global) maximality of, e.g. conflictless set of evolution rules applied simultaneously to a membrane structure which appears during the evolution process

[1] Via the representation of G–P–R machines by Gandy machines given in [5].

generated by a P system. Therefore one can construct a Gandy's machine from a G–P–R machine in an immediate way, see [5].

The NP complete problems can be solved by G–P–R machines in a polynomial time (but with an exponential number of indecomposable processors), see [5], where one constructs a G–P–R machines solving SAT problem in a polynomial time in a similar way to (families of) P systems solving this problem also in a polynomial time (cf. the pioneering Păun's paper [7]).

For all unexplained terms and notation of category theory and graph theory we refer the reader to Appendix.

Definition. A G–P–R *machine* $\mathcal{M}$ is determined by the following data:

— a finite set $\Sigma_{\mathcal{M}}$ of labels or symbols of $\mathcal{M}$,
— a skeletal set $\mathcal{S}_{\mathcal{M}}$ of finite isomorphically perfect labelled directed graphs over Σ, which are called *instantenous descriptions* of $\mathcal{M}$,
— a function $\mathcal{F}_{\mathcal{M}} : \mathcal{S}_{\mathcal{M}} \to \mathcal{S}_{\mathcal{M}}$ called the *transition function* of $\mathcal{M}$,
— a function $\mathcal{R}_{\mathcal{M}} : \mathrm{PREM}_{\mathcal{M}} \to \mathrm{CONCL}_{\mathcal{M}}$ from a finite skeletal set $\mathrm{PREM}_{\mathcal{M}}$ of finite isomorphically perfect labelled directed graphs over $\Sigma_{\mathcal{M}}$ onto a finite skeletal set $\mathrm{CONCL}_{\mathcal{M}}$ of finite isomorphically perfect labelled directed graphs over $\Sigma_{\mathcal{M}}$ such that $\mathcal{R}_{\mathcal{M}}$ determines the *set*

$$\widetilde{\mathcal{R}}_{\mathcal{M}} = \{P \vdash C \mid P \in \mathrm{PREM}_{\mathcal{M}} \text{ and } C = \mathcal{R}_{\mathcal{M}}(P)\}$$

of rewriting rules of $\mathcal{M}$ which are identified with ordered pairs $r = (P_r, C_r)$, where the graph $P_r \in \mathrm{PREM}_{\mathcal{M}}$ is the *premise* of r and the graph $C_r = \mathcal{R}_{\mathcal{M}}(P_r)$ is the *conclusion* of r,
— a subset $\mathcal{I}_{\mathcal{M}}$ of $\mathcal{S}_{\mathcal{M}}$ which is the set of *initial instantaneous descriptions of* $\mathcal{M}$.

The above data are subject of the following conditions:

1) $V(\mathcal{G}) \subseteq V(\mathcal{F}_{\mathcal{M}}(\mathcal{G}))$ for every $\mathcal{G} \in \mathcal{S}_{\mathcal{M}}$,
2) $V(\mathcal{G}) \subseteq V(\mathcal{R}_{\mathcal{M}}(\mathcal{G}))$ for every $\mathcal{G} \in \mathrm{PREM}_{\mathcal{M}}$,
3) the rewriting rules of $\mathcal{M}$ are *applicable* to $\mathcal{S}_{\mathcal{M}}$ which means that for every $\mathcal{G} \in \mathcal{S}_{\mathcal{M}}$ the set

$$\begin{aligned}\mathcal{P}\ell(\mathcal{G}) = \big\{h \mid\ & h \text{ is an embedding of labelled graphs over } \Sigma \\ & \text{with } \mathrm{dom}(h) \in \mathrm{PREM}_{\mathcal{M}} \text{ and } \mathrm{cod}(h) = \mathcal{G} \\ & \text{such that for every embedding } h' \text{ of labelled graphs over } \Sigma \\ & \text{with } \mathrm{dom}(h') \in \mathrm{PREM}_{\mathcal{M}} \text{ and } \mathrm{cod}(h') = \mathcal{G} \\ & \text{if } \mathrm{im}(h) \text{ is a labelled subgraph of } \mathrm{im}(h'), \text{ then } h = h'\big\}\end{aligned}$$

of *maximal applications* h of the rules $\mathrm{dom}(h) \vdash \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h))$ of $\mathcal{M}$ in places $\mathrm{im}(h)$ is such that the following conditions hold:
(i) $V(\mathcal{G}) = \bigcup_{h \in \mathcal{P}\ell(\mathcal{G})} V(\mathrm{im}(h))$, $E(\mathcal{G}) = \bigcup_{h \in \mathcal{P}\ell(\mathcal{G})} E(\mathrm{im}(h))$,

(ii) for all $h_1, h_2 \in \mathcal{P}\ell(\mathcal{G})$ the equation $\ell_{\mathcal{G}_{h_1}}(\dot{h}_1^{-1}(v)) = \ell_{\mathcal{G}_{h_2}}(\dot{h}_2^{-1}(v))$ holds for every $v \in V(\mathrm{im}(h_1)) \cap V(\mathrm{im}(h_2))$, where $\ell_{\mathcal{G}_{h_1}}$, $\ell_{\mathcal{G}_{h_2}}$ are the labelling functions of $\mathcal{G}_{h_1} = \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h_1))$, $\mathcal{G}_{h_2} = \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h_2))$, respectively, and $\dot{h}_1^{-1}$, $\dot{h}_2^{-1}$ are the inverses of isomorphisms induced by the embeddings h_1, h_2, respectively.

(iii) $\mathcal{F}_{\mathcal{M}}(\mathcal{G})$ is a colimit of a gluing diagram $\mathcal{D}^{\mathcal{G}}$ constructed in the following way (the construction of $\mathcal{D}^{\mathcal{G}}$ is provided by (ii)):
- the set $\mathcal{I}$ of indexes of $\mathcal{D}^{\mathcal{G}}$ is such that $\mathcal{I} = \mathcal{P}\ell(\mathcal{G}) \cup \{\Delta\}$, where $\Delta \notin \mathcal{P}\ell(\mathcal{G})$ is the center of $\mathcal{D}^{\mathcal{G}}$,
- the family $\mathcal{G}_i$ $(i \in \mathcal{I})$ of labelled graphs of $\mathcal{D}^{\mathcal{G}}$ is such that $\mathcal{G}_h = \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h))$ for every $h \in \mathcal{P}\ell(\mathcal{G})$, and $\mathcal{G}_\Delta$ is such that $V(\mathcal{G}_\Delta) = V(\mathcal{G})$, $E(\mathcal{G}_\Delta) = \varnothing$, and the labelling function $\ell_{\mathcal{G}_\Delta}$ is such that provided by (ii)
$$\ell_{\mathcal{G}_\Delta}(v) = \ell_{\mathcal{G}_h}(\dot{h}^{-1}(v))$$
for every $v \in V(\mathrm{im}(h))$ and every $h \in \mathcal{P}\ell(\mathcal{G})$, where $\dot{h}^{-1}$ is the inverse of the isomorphism $\dot{h}$ induced by the embedding h,
- the gluing conditions gl_h $(h \in \mathcal{P}\ell(\mathcal{G}))$ of $\mathcal{D}^{\mathcal{G}}$ are defined by
$$\mathrm{gl}_h = \big\{(v, \dot{h}^{-1}(v)) \mid v \in V(\mathrm{im}(h))\big\}$$
for every $h \in \mathcal{P}\ell(\mathcal{G})$, where $\dot{h}^{-1}$ is the inverse of the isomorphism $\dot{h}$ induced by embedding h,

(iv) the following equations hold:
$$V(\mathcal{F}_{\mathcal{M}}(\mathcal{G})) = \bigcup_{i \in \mathcal{I}} V(\mathrm{im}(q_i))$$
$$\text{and } E(\mathcal{F}_{\mathcal{M}}(\mathcal{G})) = \bigcup_{i \in \mathcal{I}} E(\mathrm{im}(q_i))$$
for the canonical injections $q_i : \mathcal{G}_i \to \mathcal{F}_{\mathcal{M}}(\mathcal{G})$ $(i \in \mathcal{I})$ forming a colimiting cocone of the diagram $\mathcal{D}^{\mathcal{G}}$ defined in (iii),

(v) the canonical injection $q_\Delta : \mathcal{G}_\Delta \to \mathcal{F}_{\mathcal{M}}(\mathcal{G})$ is an inclusion of labelled graphs, where Δ is the center of $\mathcal{D}^{\mathcal{G}}$ and q_Δ is an element of the colimiting cocone in (iv).

Thus $\mathcal{F}_{\mathcal{M}}(\mathcal{G})$ is the result of simultaneous application of the rules $\mathrm{dom}(h) \vdash \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h))$ in the places $\mathrm{im}(h)$ for $h \in \mathcal{P}\ell(\mathcal{G})$, where one replaces simultaneously $\mathrm{im}(h)$ by $\mathrm{im}(q_h)$ in $\mathcal{G}$ for $h \in \mathcal{P}\ell(\mathcal{G})$, respectively.

A finite sequence $\big(\mathcal{F}^i_{\mathcal{M}}(\mathcal{G})\big)_{i=0}^n$ is called a *finite computation of* $\mathcal{M}$, the number n is called the *time* of this computation, and $\mathcal{F}^n_{\mathcal{M}}(\mathcal{G})$ is called the *final instantaneous description* for this computation if

$$\mathcal{F}^0_{\mathcal{M}}(\mathcal{G}) = \mathcal{G} \in \mathcal{I}_{\mathcal{M}}, \quad \mathcal{F}^{n-1}_{\mathcal{M}}(\mathcal{G}) \neq \mathcal{F}^n_{\mathcal{M}}(\mathcal{G}), \quad \text{and } \mathcal{F}_{\mathcal{M}}(\mathcal{F}^n_{\mathcal{M}}(\mathcal{G})) = \mathcal{F}^n_{\mathcal{M}}(\mathcal{G}),$$

where $\mathcal{F}^i_{\mathcal{M}}(\mathcal{G})$ is defined inductively: $\mathcal{F}^i_{\mathcal{M}}(\mathcal{G}) = \mathcal{F}_{\mathcal{M}}\big(\mathcal{F}^{i-1}_{\mathcal{M}}(\mathcal{G})\big)$.

For a computation $\big(\mathcal{F}^i_{\mathcal{M}}(\mathcal{G})\big)_{i=0}^n$ its *space* is defined by

$$\mathrm{space}(\mathcal{M}, \mathcal{G}) = \max\{\text{the number of elements of } V(\mathcal{F}^i_{\mathcal{M}}(\mathcal{G})) \mid 0 \leq i \leq n\}$$

for $\mathcal{G} \in \mathcal{I}_{\mathcal{M}}$, where intuitively space$(\mathcal{M}, \mathcal{G})$ is understood as the size of hardware measured by the number of indecomposable processors[2] used in the computations.

Example 1 (G–P–R machine simulating the computations of a Turing machine). Let $\mathbb{T}$ be a Turing machine[3] whose alphabet Σ (including blank symbol) is disjoint with the set Q of states of $\mathbb{T}$ and let $\delta : \Sigma \times Q \to \Sigma \times Q \times \{L, 0, R\}$ be the transition of $\mathbb{T}$ with cursor directions L for "left", 0 for "stay", and R for "right". We define a *graphical instantaneous description of* $\mathbb{T}$ to be a labelled directed graph $\mathcal{G}$ over $\Sigma^0 = \Sigma \cup Q \cup \{\%, \S\}$ with $\{\%, \S\} \cap (\Sigma \cup Q) = \varnothing$ such that

- $\mathcal{G}$ is induced by some acceptable ordered triple of integers, see Appendix,
- if $\mathcal{G}$ is induced by an acceptable ordered triple (k, m, n) of integers, then $\ell_{\mathcal{G}}(-k) = \%$, $\ell_{\mathcal{G}}(0) \in Q$, $\ell_{\mathcal{G}}(n) = \S$ and $\ell_{\mathcal{G}}(j) \in \Sigma$ for every $j \in \big\{i \in V(\mathcal{G}) \mid -k < i < n \text{ and } i \neq 0\big\}$ (here m corresponds to cursor position on Turing machine tape indicated by the edge $(0, m)$).

By Lemma 4 in Appendix the set $\mathcal{S}_{\mathbb{T}}$ of all graphical instantaneous descriptions of $\mathbb{T}$ is a skeletal set of isomorphically perfect labelled graphs. Thus we define a G–P–R machine $\mathcal{M}_{\mathbb{T}}$ aimed to simulate the computations of $\mathbb{T}$ such that

- the set of instantaneous descriptions of $\mathcal{M}_{\mathbb{T}}$ is the set $\mathcal{S}_{\mathbb{T}}$ of graphical instantaneous descriptions of $\mathbb{T}$,
- the transition function $\mathcal{F}_{\mathbb{T}}$ of $\mathcal{M}_{\mathbb{T}}$ and the rewriting rules of $\mathcal{M}_{\mathbb{T}}$ are determined by the transition function δ of $\mathbb{T}$ such that if $\delta(a, q) = (a', q', R)$, then

  (f^R) if $\mathcal{G} \in \mathcal{S}_{\mathbb{T}}$ and $\mathcal{G}$ is induced by (k, m, n) such that $\ell_{\mathcal{G}}(m) = a$, $\ell_{\mathcal{G}}(0) = q$, then
    1) if $m < n-1$, then $\mathcal{F}_{\mathbb{T}}(\mathcal{G})$ is that $\mathcal{G}'$ which is induced by $(k, \widehat{m}, n)$ with $\widehat{m} = m+1$ for $m \neq -1$ and $\widehat{m} = 1$ for $m = -1$ such that $\ell_{\mathcal{G}'}(0) = q'$, $\ell_{\mathcal{G}'}(m) = a'$, and $\ell_{\mathcal{G}'}(i) = \ell_{\mathcal{G}}(i)$ for every $i \in V(\mathcal{G}) - \{0, m\}$,
    2) if $m = n-1$, then $\mathcal{F}_{\mathbb{T}}(\mathcal{G})$ is that $\mathcal{G}'$ which is induced by $(k, m+1, n+1)$ such that $\ell_{\mathcal{G}'}(0) = q'$, $\ell_{\mathcal{G}'}(m) = a'$, $\ell_{\mathcal{G}'}(n)$ is blank symbol, and $\ell_{\mathcal{G}'}(i) = \ell_{\mathcal{G}}(i)$ for every $i \in V(\mathcal{G}') - \{0, m, n\}$,

  (r^R) the rewriting rules are given by the following two schemes $\mathcal{G}_p \vdash \mathcal{G}_c$ such that

    (r_1^R) the premise $\mathcal{G}_p$ is such that $V(\mathcal{G}_p) = \{-1, 0, 1, 2\}$, $E(\mathcal{G}_p) = \mathrm{lin}[1, 2] \cup \{(0, 1)\}$, $\ell_{\mathcal{G}_p}(-1) \in \Sigma \cup \{\%\}$, $\ell_{\mathcal{G}_p}(0) = q$, $\ell_{\mathcal{G}_p}(1) = a$, $\ell_{\mathcal{G}_p}(2) \in \Sigma$, the conclusion $\mathcal{G}_c$ is such that $V(\mathcal{G}_c) = V(\mathcal{G}_p)$, $E(\mathcal{G}_c) = \mathrm{lin}[1, 2] \cup \{(0, 2)\}$, $\ell_{\mathcal{G}_c}(-1) = \ell_{\mathcal{G}_p}(-1)$, $\ell_{\mathcal{G}_c}(0) = q'$, $\ell_{\mathcal{G}_c}(1) = a'$, and $\ell_{\mathcal{G}_c}(2) = \ell_{\mathcal{G}_p}(2)$.

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[2] The indecomposable processors coincide with urelements appearing in those Gandy machines which represent G–P–R machines in [5].

[3] For the definition and unexplained terms we refer the reader to, e.g., [6].

(r_2^R) the premise $\mathcal{G}_p$ is such that $V(\mathcal{G}_p) = \{-1, 0, 1, 2\}$, $E(\mathcal{G}_p) = \mathrm{lin}[1, 2] \cup \{(0, 1)\}$, $\ell_{\mathcal{G}_p}(-1) \in \Sigma \cup \{\%\}$, $\ell_{\mathcal{G}_p}(0) = q$, $\ell_{\mathcal{G}_p}(1) = a$, $\ell_{\mathcal{G}_p}(2) = \S$, the conclusion $\mathcal{G}_c$ is such that $V(\mathcal{G}_c) = \{-1, 0, 1, 2, 3\}$, $E(\mathcal{G}_c) = \mathrm{lin}[1, 3] \cup \{(0, 2)\}$, $\ell_{\mathcal{G}_c}(-1) = \ell_{\mathcal{G}_p}(-1)$, $\ell_{\mathcal{G}_c}(0) = q'$, $\ell_{\mathcal{G}_c}(1) = a'$, $\ell_{\mathcal{G}_c}(2)$ is blank symbol, and $\ell_{\mathcal{G}_c}(3) = \S$.

For the cases of equations $\delta(a, q) = (a', q', 0)$ and $\delta(a, q) = (a', q', L)$ the values $\mathcal{F}_{\mathbb{T}}(\mathcal{G})$ and the rewriting rules are defined in a similar way, where, e.g., the counterpart of (f_2^R) for $\delta(a, q) = (a', q', L)$ is:

(f_2^L) if $1 = m = k$ or $-k + 1 = m \neq 0$, then $\mathcal{F}_{\mathbb{T}}(\mathcal{G})$ is that $\mathcal{G}'$ which is induced by $(k + 1, -k, n)$ such that $\ell_{\mathcal{G}'}(-k - 1) = \%$, $\ell_{\mathcal{G}'}(-k)$ is blank symbol, $\ell_{\mathcal{G}'}(0) = q'$, $\ell_{\mathcal{G}'}(m) = a'$, and $\ell_{\mathcal{G}'}(i) = \ell_{\mathcal{G}}(i)$ for all $i \in V(\mathcal{G}') - \{-k - 1, -k, 0, m\}$.

The versions of the above rules $\mathcal{G}_p \vdash \mathcal{G}_c$ for both $\mathcal{G}_p$ and $\mathcal{G}_c$ completed by the loop (i, i) for a unique $i \in V(\mathcal{G}_p)$ with $\ell_{\mathcal{G}_p}(i) \notin \{\%, \S\} \cup Q$ are also necessary. The identity rules $\mathcal{G} \vdash \mathcal{G}$ are also necessary, where $\mathcal{G}$ is of the following two forms:

(id_1) $V(\mathcal{G}) = \{0, 1\}$, $E(\mathcal{G}) = \{(0, 1)\}$, $\{\ell_{\mathcal{G}}(0), \ell_{\mathcal{G}}(1)\} \subset \Sigma^0 - Q$,
(id_2) $V(\mathcal{G}) = \{0\}$, $E(\mathcal{G}) = \{(0, 0)\}$, $\ell_{\mathcal{G}}(0) \in \Sigma$.

There is no other rewriting rule of $\mathcal{M}_{\mathbb{T}}$ than that described by the above schemes.

Since the graphical instantaneous descriptions of a Turing machine $\mathbb{T}$ coincide with the usual instantaneous descriptions of $\mathbb{T}$ or configurations of $\mathbb{T}$ as in [6], the G–P–R machine $\mathcal{M}_{\mathbb{T}}$ simulates the computations of $\mathbb{T}$ due to definition of $\mathcal{F}_{\mathbb{T}}$.

Example 2 (G–P–R machine simulating the computations of certain Boolean circuits). We define a *disjunctive circuit* G–P–R *machine* $\mathcal{M}_{\mathrm{circ}}$ which is aimed to simulate computations of certain tree like Boolean circuits such that

- the set $\mathcal{S}_{\mathrm{circ}}$ of instantaneous descriptions of $\mathcal{M}_{\mathrm{circ}}$ is the set of those regular labelled binary trees $\mathcal{T}$ of depth greater than 3 over the set $\{\mathrm{root}, 0, 1\} \times \{\bot, 0, 1\}$ of labels, see Appendix, which satisfy the following condition
  (circ_0) for every binary string $\Gamma \in V(\mathcal{T})$ of length equal to the depth of $\mathcal{T}$ the number of elements of the set
  $$\{i \mid i \text{ is a natural number with } 0 < i \leq n \text{ such that } \ell^2_{\mathcal{T}}(\Gamma \restriction i) \neq \bot\}$$
  is not greater than 1 (thus this set may be empty), where n is the depth of $\mathcal{T}$ and if Γ is $(k_j)_{j=1}^n$ then $\Gamma \restriction i$ denotes the string $(k_j)_{j=1}^i$ which is Γ itself for $i = n$ and for $i < n$ $(k_j)_{j=1}^i$ is a shortening of Γ by cancellation of the elements $k_n, k_{n-1}, \ldots, k_{i+1}$.
- the transition function $\mathcal{F}_{\mathrm{circ}}$ of $\mathcal{M}_{\mathrm{circ}}$ is such that $\mathcal{F}_{\mathrm{circ}}(\mathcal{T})$ is the result of simultaneous application to $\mathcal{T}$ in G–P–R machine mode[4] the rewriting rules

[4] Understood that the result of simultaneous application is a colimit of the gluing diagram determined by the set $\mathcal{P}\ell(\mathcal{G})$ of maximal applications as in definition of G–P–R machine.

of $\mathcal{M}_{\mathrm{circ}}$ which do not introduce new vertices and which are given by the following three schemes $\mathcal{T}_p \vdash \mathcal{T}_c$ such that

(circ$_1$) the premise $\mathcal{T}_p$ is such that $V(\mathcal{T}_p) = \{\Lambda, 0, 00, 01\}$,
$E(\mathcal{T}_p) = \big\{(\Lambda, 0), (0, 00), (0, 01), (00, 00), (01, 01)\big\}$,
$\ell^2_{\mathcal{T}_p}(\Lambda) = \ell^2_{\mathcal{T}_p}(0) = \bot$, $\{\ell^1_{\mathcal{T}_p}(\Lambda), \ell^1_{\mathcal{T}_p}(0)\} \subseteq \{0,1\}$,
$\ell^1_{\mathcal{T}_p}(00) = 0$, $\ell^1_{\mathcal{T}_p}(01) = 1$, $\{\ell^2_{\mathcal{T}_p}(00), \ell^2_{\mathcal{T}_p}(01)\} \subseteq \{0,1\}$,
the conclusion $\mathcal{T}_c$ is such that $V(\mathcal{T}_c) = V(\mathcal{T}_p)$, $E(\mathcal{T}_c) = E(\mathcal{T}_p)$,
$\ell_{\mathcal{T}_c}(\Lambda) = \ell_{\mathcal{T}_p}(\Lambda)$, $\ell_{\mathcal{T}_c}(0) = \big(\ell^1_{\mathcal{T}_p}(0), \max\{\ell^2_{\mathcal{T}_p}(00), \ell^2_{\mathcal{T}_p}(01)\}\big)$,
$\ell_{\mathcal{T}_c}(00) = (0, \bot)$, $\ell_{\mathcal{T}_c}(01) = (1, \bot)$,

(circ$_2$) the premise $\mathcal{T}_p$ is such that $V(\mathcal{T}_p) = \{\Lambda, 0, 00, 01, 000, 001, 010, 011\}$,
$E(\mathcal{T}_p) = \big\{(\Gamma, \Gamma i) \mid \{\Gamma, \Gamma i\} \subseteq V(\mathcal{T}_p) \text{ and } i \in \{0,1\}\big\}$,
$\ell^2_{\mathcal{T}_p}(\Gamma) = \bot$ for all $\Gamma \in V(\mathcal{T}_p) - \{00, 01\}$,
$\{\ell^2_{\mathcal{T}_p}(00), \ell^2_{\mathcal{T}_p}(01)\} \subseteq \{0,1\}$, $\{\ell^1_{\mathcal{T}_p}(\Lambda), \ell^1_{\mathcal{T}_p}(0)\} \subseteq \{0,1\}$,
$\ell^1_{\mathcal{T}_p}(\Gamma i) = i$ for all $\Gamma \in \{0, 00, 01\}$ and $i \in \{0,1\}$,
the conclusion $\mathcal{T}_c$ is such that $V(\mathcal{T}_c) = V(\mathcal{T}_p)$, $E(\mathcal{T}_c) = E(\mathcal{T}_p)$,
$\ell_{\mathcal{T}_c}(\Gamma) = \ell_{\mathcal{T}_p}(\Gamma)$ for every $\Gamma \in V(\mathcal{T}_c) - \{0, 00, 01\}$,
$\ell_{\mathcal{T}_c}(0) = \big(\ell_{\mathcal{T}_p}(0), \max\{\ell^2_{\mathcal{T}_p}(00), \ell^2_{\mathcal{T}_p}(01)\}\big)$,
$\ell_{\mathcal{T}_c}(\Gamma) = \big(\ell^1_{\mathcal{T}_p}(\Gamma), \bot\big)$ for every $\Gamma \in \{00, 11\}$,

(circ$_3$) the premise $\mathcal{T}_p$ is such that $V(\mathcal{T}_p) = \{\Lambda, 0, 1, 00, 01, 10, 11\}$,
$E(\mathcal{T}_p) = \big\{(\Gamma, \Gamma i) \mid \{\Gamma, \Gamma i\} \subseteq V(\mathcal{T}_p) \text{ and } i \in \{0,1\}\big\}$,
$\ell^1_{\mathcal{T}_p}(\Gamma i) = i$ for all $\Gamma \in \{\Lambda, 0, 1\}$ and $i \in \{0,1\}$,
$\ell^2_{\mathcal{T}_p}(\Gamma) = \bot$ for every $\Gamma \in V(\mathcal{T}_p) - \{0,1\}$,
$\ell^1_{\mathcal{T}_p}(\Lambda) = \text{root}$, $\{\ell^2_{\mathcal{T}_p}(0), \ell^2_{\mathcal{T}_p}(1)\} \subseteq \{0,1\}$,
the conclusion $\mathcal{T}_c$ is such that $V(\mathcal{T}_c) = V(T_p)$, $E(T_c) = E(\mathcal{T}_p)$,
$\ell^2_{\mathcal{T}_c}(\Gamma) = \ell^2_{\mathcal{T}_p}(\Gamma)$ for every $\Gamma \in V(\mathcal{T}_c) - \{\Lambda, 0, 1\}$,
$\ell_{\mathcal{T}_c}(\Gamma) = (\ell^1_{\mathcal{T}_p}, \bot)$ for every $\Gamma \in \{0,1\}$,
and $\ell_{\mathcal{T}_c}(\Lambda) = \big(\text{root}, \max\{\ell^2_{\mathcal{T}_p}(0), \ell^2_{\mathcal{T}_p}(1)\}\big)$.

The identity rules $\mathcal{T} \vdash \mathcal{T}$ are also necessary which are defined in a similar way as in Example 1.

There is no other rewriting rule of $\mathcal{M}_{\mathrm{circ}}$ than that described by the above schemes.

The following lemma characterizes the computations of G–P–R machine $\mathcal{M}_{\mathrm{circ}}$.

Lemma 1. *Let $\mathcal{T} \in \mathcal{S}_{\mathrm{circ}}$ be a regular labelled binary tree such that for every binary string Γ of length equal to the depth of $\mathcal{T}$ there exists a natural number i with $i > 0$ such that $\ell^2_{\mathcal{T}}(\Gamma \restriction i) \neq \bot$. Then for*

$$n = \max\big\{i \mid i \textit{ is the length of some binary string } \Gamma \in V(\mathcal{T}) \textit{ with } \ell^2_{\mathcal{T}}(\Gamma) \neq \bot\big\}$$

the value $\mathcal{F}^n_{\mathrm{circ}}(\mathcal{T})$ is that regular labelled tree $\mathcal{T}'$ which is such that $V(\mathcal{T}') = V(\mathcal{T})$, $E(\mathcal{T}') = E(\mathcal{T})$, $\ell^2_{\mathcal{T}'}(\Gamma) = \bot$ for all $\Gamma \in V(\mathcal{T}') - \{\Lambda\}$ and $\ell^2_{\mathcal{T}'}(\Lambda) = \max\big\{\ell^2_{\mathcal{T}}(\Gamma) \mid \Gamma \in V(\mathcal{T}')$ and $\ell^2_{\mathcal{T}}(\Gamma) \neq \bot\big\}$, where $\mathcal{F}^n_{\mathrm{circ}}(\mathcal{T})$ is defined inductively by $\mathcal{F}^1_{\mathrm{circ}}(\mathcal{T}) = \mathcal{F}_{\mathrm{circ}}(\mathcal{T})$ and $\mathcal{F}^n_{\mathrm{circ}}(\mathcal{T}) = \mathcal{F}_{\mathrm{circ}}(\mathcal{F}^{n-1}_{\mathrm{circ}}(\mathcal{T}))$.

Example 3 (assembly of binary trees). The set $\mathcal{S}_{\text{tree}}$ of labelled directed graphs $\mathcal{T}_n^j$ for natural numbers j, n with $0 \leq j \leq 3 \cdot n - 3$ (for the definition of $\mathcal{T}_n^j$ see Appendix) is the set of instantaneous descriptions of a G–P–R machine $\mathcal{M}_{\text{tree}}$ whose transition function $\mathcal{F}_{\text{tree}}$ is defined by

$$\mathcal{F}_{\text{tree}}(\mathcal{T}_n^j) = \mathcal{T}_n^{j+1} \quad (j \geq 0,\ n \geq 0).$$

The rewriting rules of $\mathcal{M}_{\text{tree}}$ are given by the modified versions of the schemes $(\dot{r}_1)$, $(\dot{r}_{10})$, $(\dot{r}_{11})$, $(\dot{r}_{12})$ (see Appendix), and the identity rules, where, e.g., the claimed modification of $(\dot{r}_1)$ is obtained by restricting the scheme to vertices 1, 2. By Lemmas 6 and 7 in Appendix the machine $\mathcal{M}_{\text{tree}}$ starting with $\mathcal{T}_n^0$ stops after $3 \cdot n - 3$ steps with $\mathcal{T}_n^{3 \cdot n - 3}$ as the final result which is isomorphic to a regular labelled binary tree of depth n ($n \geq 0$). The machine $\mathcal{M}_{\text{tree}}$ is inspired by the similar constructs in [8], where the membrane division rules correspond to the above rewriting rules of $\mathcal{M}_{\text{tree}}$.

3 Randomized Gandy-Păun-Rozenberg machines and NP complete problems

We propose an open, with some loss of precision, definition of randomized G–P–R machines to provide various their applications like applications in modelling systems solving NP complete problems shown in the present paper and the future applications in simulation of quantum computer computations by randomized systems.

Definition. By a *randomized* G–P–R *machine* we understand a G–P–R machine whose initial instantaneous descriptions contain certain configurations or structures chosen at random. These randomly chosen configurations or structures cause an uncertainty of the final result of machine computations which is measured by an error probability.

The randomly chosen configurations or structures will be described more precisely for particular applications, respectively.

We use the following basic concepts for randomization of G–P–R machines discussed in the present paper.

Definitions. By a *ground random ternary sequence* of length n we understand a finite sequence $\Theta = (\sigma_i^\Theta)_{i=1}^n$ of digits 0, 1, and a symbol @ with all elements $\sigma_i^\Theta = \{0, 1, @\}$ chosen at random and with $k > 0$ occurrences of @ in Θ such that possibly $k = f(n)$ for some f satisfying $\lim\limits_{n\to\infty} \frac{f(n)}{n} = 0$.

A ground random ternary sequence $\Theta = (\sigma_i^\Theta)_{i=1}^n$ with k occurrences of @ in Θ gives rise to 2^k binary strings $\Gamma = (\sigma_i^\Gamma)_{i=1}^n$ of length n which are obtained from Θ by replacing the occurrences of @ in Θ by the occurrences of digits 0 or 1, i.e., $\sigma_i^\Gamma \in \{0, 1\}$ for $\sigma_i^\Theta = @$ and $\sigma_i^\Gamma = \sigma_i^\Theta$ for $\sigma_i^\Theta \in \{0, 1\}$. These 2^k binary strings Γ obtained from Θ, called *binary strings generated* by Θ, are random binary strings with randomness inherited from Θ.

A ground random ternary sequence Θ of length n with k occurrences of @ can be constructed from e.g. two random binary strings Γ and Γ' of length n and $n-k$, respectively, such that @ is the i-th element of Θ iff 1 is the i-th element of Γ and deleting all occurrences of @ in Θ yields Γ'.

For all unexplained terms of logic and computational complexity theory, including Turing machines and the formulation of SAT and 3-SAT problems, we refer the reader to [6].

Example 4 (a randomized G–P–R machine solving 3-SAT problem in a polynomial time). We use a Turing machine $\dot{\mathbb{T}}$ such that for every formula φ in a disjunctive normal form as in 3-SAT problem and every truth assignment T for variables of φ the machine decides in the time $\leq n^{k_0}$ whether φ is valid for T, where the ordered pair (φ, T) is an input for $\dot{\mathbb{T}}$ from which the machine begins the computation, k_0 is some constant natural number, and n is the number of variables occurring in φ. We claim for $\dot{\mathbb{T}}$ that:

(A) if n is the number of variables occurring in φ, then any truth assignment T for variables of φ is represented by that binary string Γ of length n in the machine tape which is such that if the value "True" is assigned to the i-th variable of φ, then 1 is the i-th element of Γ, otherwise the i-th element of Γ is 0,

(A′) Turing machine $\dot{\mathbb{T}}$ is constructed from some simpler three-string or three-tape Turing machine 3-$\mathbb{T}$ according to the general construction in the proof of Theorem 2.1, p. 30 of [6], where the first tape of 3-$\mathbb{T}$ is an output tape, the second tape is an input tape containing some presentation of a truth assignment, and the third tape is an input tape containing some presentation of a formula. The machine 3-$\mathbb{T}$ reads only its input tapes and does not move its head or cursor to the left or right on output tape.

(B) for the G–P–R machine $\mathcal{M}_{\dot{\mathbb{T}}}$ simulating the computations of $\dot{\mathbb{T}}$ if we have that

- (b_1) $\mathcal{G}_{\varphi,\Gamma}$ is that initial instantaneous description of $\mathcal{M}_{\dot{\mathbb{T}}}$ which coincides with the initial instantaneous description or initial configuration for input (φ, T) with T represented by the binary string Γ in the machine tape as in (A),
- (b_2) $\mathcal{G} = \mathcal{F}^q_{\dot{\mathbb{T}}}(\mathcal{G}_{\varphi,\Gamma})$ is the final instantaneous description of $\mathcal{M}_{\dot{\mathbb{T}}}$ for the case of the final or halting state "stop" reached by $\dot{\mathbb{T}}$ after q steps of computation starting with input (φ, T) with T related to Γ as in (A), where $\mathcal{F}_{\dot{\mathbb{T}}}$ is the transition function of $\mathcal{M}_{\dot{\mathbb{T}}}$ and $\mathcal{F}^q_{\dot{\mathbb{T}}}(\mathcal{G}_{\varphi,\Gamma})$ is inductively defined: $\mathcal{F}^1_{\dot{\mathbb{T}}}(\mathcal{G}_{\varphi,\Gamma}) = \mathcal{F}_{\dot{\mathbb{T}}}(\mathcal{G}_{\varphi,\Gamma})$ and $\mathcal{F}^q_{\dot{\mathbb{T}}}(\mathcal{G}_{\varphi,\Gamma}) = \mathcal{F}_{\dot{\mathbb{T}}}(\mathcal{F}^{q-1}_{\dot{\mathbb{T}}}(\mathcal{G}_{\varphi,\Gamma}))$,

then

- (b_1') $\mathcal{G}_{\varphi,\Gamma}$ is a labelled directed graph induced by an acceptable ordered triple $(1, 1, m^0_\varphi)$ providing natural numbers m^-_φ, m^+_φ with $1 < m^-_\varphi < m^+_\varphi < m^0_\varphi$ and $m^+_\varphi - m^-_\varphi = n$, such that $\ell_{\mathcal{G}_{\varphi,\Gamma}}(m^-_\varphi + j)$ is the j-th element of Γ for every j with $1 \leq j \leq n$, where the numbers m^-_φ, m^+_φ are determined by the construction of $\dot{\mathbb{T}}$ from 3-$\mathbb{T}$ such that the essential content of the second tape of 3-$\mathbb{T}$, i.e. Γ itself, is written in the n squares $(m^-_\varphi + 1)$-th, …, m^+_φ-th, respectively, of the tape of $\dot{\mathbb{T}}$,

(b_2') $\mathcal{G}$ is a labelled graph induced by some acceptable triple $(1, 1, m_\varphi^0)$ of integers such that $\ell_\mathcal{G}(0)$ is the final state "stop" and $\ell_\mathcal{G}(1) = 1$ if φ is valid for the truth assignment represented by Γ, otherwise $\ell_\mathcal{G}(1) = 0$.

The shape of formulas in a disjunctive normal form in 3-SAT problem (it suffices to consider formulas of $n > 3$ variables which are disjunctions of $2^3 \cdot \binom{n}{3}$ nonrepetitive clauses, each conjunction of three literals containing different variables) provides that the machine 3-$\mathbb{T}$ reaches the final state in the time not greater than $2^3 \cdot n^5$ steps for a formula of n variables, hence by Theorem 2.1, p. 30 of [6] the claimed machine $\dot{\mathbb{T}}$ reaches the final state in the time not greater than $2^6 \cdot n^{10}$ steps for a formula of n variables.

For a function f defined on and valued in the set of natural numbers with $\lim\limits_{n\to\infty} \frac{f(n)}{n} = 0$ we outline a construction of a randomized G–P–R machine $\mathcal{M}^f_{\text{3-SAT}}$ aimed to solve 3-SAT problem in a polynomial time.

We introduce now those classes of labelled directed graphs over $\Sigma^\bullet$ which we then use to define instantaneous descriptions of $\mathcal{M}^f_{\text{3-SAT}}$ for $\Sigma^\bullet$ equal to a disjoint union $\Sigma \dot\cup Q \dot\cup \{q_0, q_1, q_2, q_3, q_4\} \dot\cup \{\S, @\} \dot\cup (\{0, 1, \bot, \text{root}\} \times \{0, 1, \bot\})$, where Σ and Q are the alphabet and the set of states of the Turing machine $\dot{\mathbb{T}}$, respectively, where $\Box \in \Sigma$ is used to denote empty square of the tape of the machine $\dot{\mathbb{T}}$ and the states q_{start}, 'stop' of $\dot{\mathbb{T}}$ are the starting state and the halting state of $\dot{\mathbb{T}}$, respectively.

For natural numbers j, m, m', n with $0 \le j \le 3 \cdot f(n) - 3$, $1 < m < m'$, $n > 3$, and for characteristic function δ_n of the subset $\{3 \cdot q - 2 \mid 0 < q < f(n)\} \cup \{3 \cdot f(n) - 3\}$ of the set $\{0, 1, \ldots, 3 \cdot f(n) - 3\}$ we define a class $\mathcal{K}_{n,j}^{\delta_n(j)\cdot m, m'}$ to be the class of labelled directed graphs $\mathcal{G}$ over $\Sigma^\bullet$ which are such that

- $V(\mathcal{G}) = \left(T^j_{f(n)} \times \{(0,0)\}\right) \cup \bigcup\limits_{(i,f(n)) \in T^j_{f(n)}} \left(V^{(i,f(n))} \times \{(i, f(n))\}\right)$ for

$$V^{(i,f(n))} = \begin{cases} \{0, 1, \ldots, m'\} & \text{if } \ell^1_{\mathcal{T}^j_{f(n)}}\big((i, f(n))\big) \in \{0, \bot\} \text{ or } \delta_n(j) = 0, \\ \{1, \ldots, m\} & \text{if } \ell^1_{\mathcal{T}^j_{f(n)}}\big((i, f(n))\big) = 1 \text{ and } \delta_n(j) = 1, \end{cases}$$

  for $T^j_{f(n)}$ and $\mathcal{T}^j_{f(n)}$ see Appendix,
- $E(\mathcal{G}) = \big\{\big((v, (0,0)), (v', (0,0))\big) \mid (v, v') \in E(\mathcal{T}^j_{f(n)})\big\} \cup$ $\bigcup_{(i,f(n)) \in T^j_{f(n)}} \Big(\big\{\big(((i, f(n)), (0,0)), (1, (i, f(n)))\big)\big\} \cup$ $\big\{\big((q, (i, f(n))), (q+1, (i, f(n)))\big) \mid q > 0 \text{ and } \{q, q+1\} \subset V^{(i,f(n))}\big\}\Big) \cup X_j$, where X_j is determined by j in the following way:
  (I) if $\delta_n(j) = 0$ then

$$X_j = \big\{\big((0, (i, f(n))), (1, (i, f(n)))\big) \mid (i, f(n)) \in T^j_{f(n)}\big\},$$

(II) if $\delta_n(j) = 1$ and $j < 3 \cdot f(n) - 3$, then there exists a natural number p with $1 \leq p < m$ such that

$$X_j = K_p^{\nabla} = \bigcup \Big\{ \nabla^n_{i,p,z} \mid \ell_{\mathcal{T}^j_{f(n)}}((i, f(n))) = 0,\ \ell_{\mathcal{T}^j_{f(n)}}((z, f(n))) = 1 \text{ and } \{(v, (i, f(n))), (v, (z, f(n)))\} \subset E(\mathcal{T}^j_{f(n)}) \text{ for some } v \Big\}$$

where $\nabla^n_{i,p,z}$ is such that

$$\nabla^n_{i,p,z} = \big\{ \big((p, (i, f(n))), (p, (z, f(n)))\big), \big((0, (i, f(n))), (p, (i, f(n)))\big), \big((0, (i, f(n))), (p, (z, f(n)))\big) \big\}.$$

(III) if $j = 3 \cdot f(n) - 3$, then one of the following conditions holds:

(III′) there exists a natural number p with $1 \leq p < m$ such that $X_j = K_p^{\nabla}$,

(III″) $m = m'$ and there exists a function

$$g : \{(i, f(n)) \mid (i, f(n)) \in T^{3 \cdot f(n) - 3}_{f(n)}\} \to \{1, \ldots, m'\}$$

such that

$$X_j = \big\{ \big((0, (i, f(n))), (g((i, f(n))), (i, f(n)))\big) \mid (i, f(n)) \in T^{3 \cdot f(n) - 3}_{f(n)} \big\}.$$

For a formula φ in a disjunctive normal form of $n > 3$ variables and a ground random ternary sequence $\Theta = (\sigma_i^{\Theta})_{i=1}^n$ with $f(n)$ occurrences of @ in Θ we define an initial instantaneous description $\mathcal{G}^0_{\varphi,\Theta}$ of $\mathcal{M}^f_{\text{3-SAT}}$ to be a labelled directed graph belonging to $\mathcal{K}^{0,m^0_\varphi}_{n,0}$ such that

- $\big((0, (0, f(n))), (1, (0, f(n)))\big) \in E(\mathcal{G}^0_{\varphi,\Theta})$,
- $\ell_{\mathcal{G}^0_{\varphi,\Theta}}\big((v, (0, 0))\big) = \ell_{\mathcal{T}^0_{f(n)}}(v)$ for every $v \in T^0_{f(n)}$,
- $\ell_{\mathcal{G}^0_{\varphi,\Theta}}\big((0, (0, f(n)))\big) = q_0$,
- for r with $0 < r \leq m^0_\varphi$

$$\ell_{\mathcal{G}^0_{\varphi,\Theta}}\big((r, (0, f(n)))\big) = \begin{cases} (\bot, \bot) & \text{if } r = 1, \\ \sigma_r^{\Theta} & \text{if } r = m^-_\varphi + q \text{ and } 1 \leq q \leq n, \\ \ell_{\mathcal{G}_{\varphi,\Gamma}}(r) & \text{otherwise,} \end{cases}$$

where $\mathcal{G}_{\varphi,\Gamma}$ is an initial instantaneous description of $\mathcal{M}_{\dot{\mathbb{T}}}$ for some Γ.

Thus the set $\mathcal{I}_{\mathcal{M}^f_{\text{3-SAT}}}$ of initial instantaneous descriptions of $\mathcal{M}^f_{\text{3-SAT}}$ is the set of labelled graphs of the form $\mathcal{G}^0_{\varphi,\Theta}$ for some φ, Θ as above, where the values $\ell_{\mathcal{G}^0_{\varphi,\Theta}}\big((r, (0, f(n)))\big)$ $(m^-_\varphi < r \leq m^+_\varphi)$ form a randomly chosen configuration in the initial instantaneous description $\mathcal{G}^0_{\varphi,\Theta}$ which makes $\mathcal{M}^f_{\text{3-SAT}}$ a randomized G–P–R machine.

For a formula φ and a ground random ternary sequence Θ as above we define *instantaneous descriptions of* $\mathcal{M}^f_{\text{3-SAT}}$ *in assembly phase* to be labelled directed graphs $\mathcal{G}^k_{\varphi,\Theta}$ belonging to $\mathcal{K}^{\delta_n(j)\cdot m, m^0_\varphi}_{n,j}$ and defined inductively such that $\mathcal{G}^k_{\varphi,\Theta}$ is the result of simultaneous application to $\mathcal{G}^{k-1}_{\varphi,\Theta}$ in G–P–R machine mode the rewriting rules given by the schemes $(\dot{r}_1)$–$(\dot{r}_{12})$ presented in Appendix, and the identity rules defined in a similar way as in Section 1.

For a formula φ and a ground random ternary sequence Θ as above we define *instantaneous descriptions of* $\mathcal{M}^f_{\text{3-SAT}}$ *in computation phase* to be labelled directed graphs $\dot{\mathcal{G}}^k_{\varphi,\Theta}$ $(k \geq 0)$ defined inductively such that

— $\dot{\mathcal{G}}^0_{\varphi,\Theta} \in \mathcal{I}^{\text{comp}}_{\varphi,\Theta} = \big\{\mathcal{G}^k_{\varphi,\Theta} \mid k > 0 \text{ and } \ell_{\mathcal{G}^k_{\varphi,\Theta}}\big((0,(i,f(n)))\big) = q_{\text{start}} \text{ for some } i\big\}$,

— $\dot{\mathcal{G}}^k_{\varphi,\Theta}$ is the result of simultaneous application to $\dot{\mathcal{G}}^{k-1}_{\varphi,\Theta}$ in G–P–R machine mode the rules of $\mathcal{M}_{\dot{\mathbb{T}}}$ with % replaced by the elements of $\{0,1\} \times \{\bot\}$ and $\mathcal{M}_{\text{circ}}$, and the following new rules given by the scheme

$$\mathcal{G}_p \vdash \mathcal{G}_c, \tag{$*$}$$

where the premise $\mathcal{G}_p$ is such that
$V(\mathcal{G}_p) = \{0,1,2,3,4\}$,
$E(\mathcal{G}_p) = \big\{(i,i+1) \mid \{i,i+1\} \subseteq V(\mathcal{G}_p) - \{0\}\big\} \cup \{(0,3),(2,2)\}$,
$\ell_{\mathcal{G}_p}(i) \in \{0,1,\bot\} \times \{\bot\}$ for every $i \in \{1,2\}$, $\ell_{\mathcal{G}_p}(3) \in \{0,1\}$, $\ell_{\mathcal{G}_p}(0) =$ "stop" $\in Q$, $\ell_{\mathcal{G}_p}(4) \in \Sigma$,
the conclusion $\mathcal{G}_c$ is such that $V(\mathcal{G}_c) = V(\mathcal{G}_p)$, $E(\mathcal{G}_c) = E(\mathcal{G}_p)$,
$\ell_{\mathcal{G}_c}(i) = \ell_{\mathcal{G}_p}(i)$ for every $i \in \{1,4\}$, $\ell_{\mathcal{G}_c}(0) = \ell_{\mathcal{G}_c}(3) = \bot$,
$\ell_{\mathcal{G}_c}(2) = \big(\ell^1_{\mathcal{G}_p}(2), \ell_{\mathcal{G}_p}(3)\big)$, where $\ell^1_{\mathcal{G}_p}(2)$ is such that $\ell_{\mathcal{G}_p}(2) = (\ell^1_{\mathcal{G}_p}(2), \bot)$.

Thus the set $\mathcal{S}_{\mathcal{M}^f_{\text{3-SAT}}}$ of instantaneous descriptions of $\mathcal{M}^f_{\text{3-SAT}}$ is the union of the set of instantaneous descriptions of $\mathcal{M}^f_{\text{3-SAT}}$ in assembly phase and the set of instantaneous descriptions of $\mathcal{M}^f_{\text{3-SAT}}$ in computation phase. The transition function $\mathcal{F}^f_{\text{3-SAT}}$ of $\mathcal{M}^f_{\text{3-SAT}}$ is such that $\mathcal{F}^f_{\text{3-SAT}}(\mathcal{G}^m_{\varphi,\Theta}) = \mathcal{G}^{m+1}_{\varphi,\Theta}$ with $\mathcal{G}^m_{\varphi,\Theta} \notin \mathcal{I}^{\text{comp}}_{\varphi,\Theta}$, and $\mathcal{F}^f_{\text{3-SAT}}(\dot{\mathcal{G}}^m_{\varphi,\Theta}) = \dot{\mathcal{G}}^{m+1}_{\varphi,\Theta}$.

The set of rewriting rules of $\mathcal{M}^f_{\text{3-SAT}}$ does not contain any other rule than these introduced above for $\mathcal{M}^f_{\text{3-SAT}}$.

Theorem. *Let f be a computable function such that $\lim\limits_{n\to\infty} \frac{f(n)}{n} = 0$ and let p_φ be an estimation of the probability that a formula φ as in* 3-SAT *problem is valid for a given truth assignment. Then the* G–P–R *machine* $\mathcal{M}^f_{\text{3-SAT}}$ *solves* 3-SAT *problem in a polynomial time with subexponential number of indecomposable processors determined by f and with error probability estimated by $(p_\varphi)^{(2^{f(n)})}$ where n is the number of variables occurring in φ.*

Proof. The proof is contained in (or can be extracted from) the following description of the computation of G–P–R machine $\mathcal{M}^f_{\text{3-SAT}}$.

The computation of $\mathcal{M}^f_{\text{3-SAT}}$ consists of two phases: the assembly phase preceding the computation phase such that one splits $\mathcal{M}^f_{\text{3-SAT}}$ into two G–P–R machines $\mathcal{M}^{f,1}_{\text{3-SAT}}$ and $\mathcal{M}^{f,2}_{\text{3-SAT}}$ corresponding to these phases, respectively, in the following way:

— the machine $\mathcal{M}^{f,1}_{\text{3-SAT}}$ corresponding to assembly phase and determined by the rules $(\dot{r}_1)$–$(\dot{r}_{12})$ begins its computation with the initial instantaneous description of $\mathcal{M}^{f}_{\text{3-SAT}}$ itself and assembles initial instantaneous descriptions of the machine $\mathcal{M}^{f,2}_{\text{3-SAT}}$ by assembly some trees (see the rules $(\dot{r}_1)$, $(\dot{r}_{10})$–$(\dot{r}_{12})$) and copying appropriate subgraphs (see the rules $(\dot{r}_2)$–$(\dot{r}_9)$),

— the machine $\mathcal{M}^{f,2}_{\text{3-SAT}}$ corresponding to computation phase and determined by the rules of $\mathcal{M}_{\dot{\mathbb{T}}}$ with % replaced by the elements of $\{0,1\} \times \{\perp\}$, $\mathcal{M}_{\text{circ}}$, and the new rules $(*)$ introduced in the definition of $\mathcal{M}^{f}_{\text{3-SAT}}$ continues the computation to reach the final result of the computation of $\mathcal{M}^{f}_{\text{3-SAT}}$ itself, where the computation of $\mathcal{M}^{f,2}_{\text{3-SAT}}$ consists of the following two subphases:

- a phase of simultaneous simulation of computations of appropriate number of copies of the Turing machine $\dot{\mathbb{T}}$ which precedes the following phase
- a phase of simulation of computation of some Boolean circuit.

More precisely, for a given formula φ (as in 3-SAT problem) of $n > 3$ variables and a ground random ternary sequence Θ of length n with $f(n)$ occurrences of @ in Θ the machine $\mathcal{M}^{f,1}_{\text{3-SAT}}$ starting with the initial instantaneous description $\mathcal{G}^0_{\varphi,\Theta}$ assembles instantaneous description $\dot{\mathcal{G}}^0_{\varphi,\Theta}$ which is an initial instantaneous description of $\mathcal{M}^{f,2}_{\text{3-SAT}}$. Here $\dot{\mathcal{G}}^0_{\varphi,\Theta}$ contains, coded in some way, those $2^{f(n)}$ random binary strings of length n which are generated by Θ and represent $2^{f(n)}$ randomly chosen truth assignments for φ, respectively. Then $\mathcal{M}^{f,2}_{\text{3-SAT}}$ simultaneously simulates the computations of $2^{f(n)}$ copies of $\dot{\mathbb{T}}$, where $2^{f(n)}$ truth assignments for φ represented by $2^{f(n)}$ randomly chosen binary strings generated by Θ are the inputs together with φ for these $2^{f(n)}$ copies of $\dot{\mathbb{T}}$, respectively.

Then Boolean circuit part of $\mathcal{M}^{f}_{\text{3-SAT}}$ simulates the computation of tree-like Boolean circuit $\mathcal{C}$ of $2^{f(n)}$ input gates (see Lemma 1), where the underlying graph of $\mathcal{C}$ is a tree of depth $f(n)$ and all non-input gates of $\mathcal{C}$ are OR gates. The $2^{f(n)}$ input gates of $\mathcal{C}$ receive those inputs which are the output results of the computations of the above $2^{f(n)}$ copies of $\dot{\mathbb{T}}$, respectively. Here each input gate g is associated with that copy $\dot{\mathbb{T}}_g$ of $\dot{\mathbb{T}}$ for which g is connected with that unique vertex i of the final graphical instantaneous description of $\dot{\mathbb{T}}_g$ for which $(0, i)$ is an edge of this final graphical instantaneous description and i is labelled by the output result of $\dot{\mathbb{T}}_g$ with 0 labelled by the final or halting state of $\dot{\mathbb{T}}_g$. The inputs of $\mathcal{C}$ are simultaneously processed by $\mathcal{C}$ to give the output result in the root of the underlying graph of $\mathcal{C}$. The output result contained in the root yields an answer (with the error probability estimated in the Theorem) to a question whether there exists a truth assignment for φ such that φ is valid for this assignment. Therefore $\mathcal{M}^{f}_{\text{3-SAT}}$ solves 3-SAT problem in a polynomial time. □

Concluding remarks

For a formula φ of n variables as in 3-SAT problem and a ground random ternary sequence Θ of length n with $f(n)$ occurrences of @ in Θ the G–P–R machine $\mathcal{M}^f_{3\text{-SAT}}$ simultaneously simulates the computations of $2^{f(n)}$ copies of the Turing machine $\dot{\mathbb{T}}$ for $2^{f(n)}$ randomly chosen instances of input data (i.e., the $2^{f(n)}$ randomly chosen truth assignments[5] for φ), respectively, where these randomly chosen instances of input data are assembled in a polynomial time by the machine $\mathcal{M}^f_{3\text{-SAT}}$. Thus the machine $\mathcal{M}^f_{3\text{-SAT}}$ shows that a randomized parallelization of computations or a parallelized randomization of these computations is possible to solve 3-SAT problem in a polynomial time. The assembly of $2^{f(n)}$ of randomly chosen instances of input data by $\mathcal{M}^f_{3\text{-SAT}}$ coincides with some simultaneous random choice of these $2^{f(n)}$ instances of input data. Thus the parallelized randomized is here a spatial randomized despite to a temporal or sequential randomized realized by repeating sequentially, i.e. step by step, an experiment consisting of a random choice of a single instance of input data then processed by, e.g., a single Turing machine, where the repeating of the experiment decreases an error probability.

We point out that the initial instantaneous descriptions of $\mathcal{M}^f_{3\text{-SAT}}$ are of the size depending linearly on the size of input data of a formula and a truth assignment.

The machine $\mathcal{M}^f_{3\text{-SAT}}$ is a biologically inspired construct, according to the ideas contained in [8], which illustrates a capability of (self)assembling of a distributed system which then realizes a process of massively parallel computation. One can include this capability to a paradigm of biologically inspired computing.

Appendix. Graph-theoretical and category-theoretical preliminaries

A [*finite*] *labelled directed graph over* a set Σ of labels is defined to be an ordered triple $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}), \ell_{\mathcal{G}})$, where $V(\mathcal{G})$ is a [finite] *set of vertices* of $\mathcal{G}$, $E(\mathcal{G})$ is a subset of $V(\mathcal{G}) \times V(\mathcal{G})$ called the *set of edges* of $\mathcal{G}$, and $\ell_{\mathcal{G}}$ is a function from $V(\mathcal{G})$ into Σ called the *labelling function* of $\mathcal{G}$. We drop the adjective 'directed' if there is no risk of confusion.

A *homomorphism of a labelled directed graph* $\mathcal{G}$ *over* Σ *into a labelled directed graph* $\mathcal{G}'$ *over* Σ is an ordered triple $(\mathcal{G}, \mathrm{h} : V(\mathcal{G}) \to V(\mathcal{G}'), \mathcal{G}')$ such that h is a function from $V(\mathcal{G})$ into $V(\mathcal{G}')$ which satisfies the following conditions:

(H_1) $(v, v') \in E(\mathcal{G})$ implies $(\mathrm{h}(v), \mathrm{h}(v')) \in E(\mathcal{G}')$ for all $v, v' \in V(\mathcal{G})$,
(H_2) $\ell_{\mathcal{G}'}(\mathrm{h}(v)) = \ell_{\mathcal{G}}(v)$ for every $v \in V(\mathcal{G})$.

If a triple $h = (\mathcal{G}, \mathrm{h} : V(\mathcal{G}) \to V(\mathcal{G}'), \mathcal{G}')$ is a homomorphism of a labelled directed graph $\mathcal{G}$ over Σ into a labelled directed graph $\mathcal{G}'$ over Σ, we denote this

[5] In different way than e.g. in [1].

triple by $h : \mathcal{G} \to \mathcal{G}'$, we write $\mathrm{dom}(h)$ and $\mathrm{cod}(h)$ for $\mathcal{G}$ and $\mathcal{G}'$, respectively, according to category theory convention, and we write $h(v)$ for the value $\mathrm{h}(v)$.

A homomorphism $h : \mathcal{G} \to \mathcal{G}'$ of labelled directed graphs over Σ is an *embedding of $\mathcal{G}$ into $\mathcal{G}'$*, denoted by $h : \mathcal{G} \rightarrowtail \mathcal{G}'$, if the following condition holds:

(E) $h(v) = h(v')$ implies $v = v'$ for all $v, v' \in V(\mathcal{G})$.

An embedding $h : \mathcal{G} \rightarrowtail \mathcal{G}'$ of labelled directed graphs $\mathcal{G}, \mathcal{G}'$ over Σ is an *inclusion of $\mathcal{G}$ into $\mathcal{G}'$*, denoted by $h : \mathcal{G} \hookrightarrow \mathcal{G}'$, if the following holds:

(I) $h(v) = v$ for every $v \in V(\mathcal{G})$.

We say that a labelled directed graph $\mathcal{G}$ over Σ is a *labelled subgraph* of a labelled directed graph $\mathcal{G}'$ over Σ if there exists an inclusion $h : \mathcal{G} \hookrightarrow \mathcal{G}'$ of labelled directed graphs $\mathcal{G}, \mathcal{G}'$ over Σ.

For an embedding $h : \mathcal{G} \rightarrowtail \mathcal{G}'$ of labelled directed graphs $\mathcal{G}, \mathcal{G}'$ over Σ we define the *image* of h, denoted by $\mathrm{im}(h)$, to be a labelled directed graph $\widehat{\mathcal{G}}$ over Σ such that $V(\widehat{\mathcal{G}}) = \big\{h(v) \mid v \in V(\mathcal{G})\big\}$, $E(\widehat{\mathcal{G}}) = \big\{(h(v), h(v')) \mid (v, v') \in E(\mathcal{G})\big\}$, and the labelling function $\ell_{\hat{\mathcal{G}}}$ of $\widehat{\mathcal{G}}$ is the restriction of the labelling function $\ell_{\mathcal{G}'}$ of $V(\mathcal{G}')$ to the set $V(\widehat{\mathcal{G}})$, i.e., $\ell_{\hat{\mathcal{G}}}(v) = \ell_{\mathcal{G}'}(v)$ for every $v \in V(\widehat{\mathcal{G}})$.

A homomorphism $h : \mathcal{G} \to \mathcal{G}'$ of labelled directed graphs over Σ is an *isomorphism* of $\mathcal{G}$ into $\mathcal{G}'$ if there exists a homomorphism $h^{-1} : \mathcal{G}' \to \mathcal{G}$ of labelled directed graphs over Σ, called the inverse of h, such that the following conditons hold:

(Iz_1) $h^{-1}(h(v)) = v$ for every $v \in V(\mathcal{G})$,
(Iz_2) $h(h^{-1}(v)) = v$ for every $v \in V(\mathcal{G}')$.

We say that a labelled directed graph $\mathcal{G}$ over Σ *is isomorphic* to a labelled directed graph $\mathcal{G}'$ over Σ if there exists an isomorphism $h : \mathcal{G} \to \mathcal{G}'$ of labelled graphs $\mathcal{G}, \mathcal{G}'$ over Σ.

For an embedding $h : \mathcal{G} \rightarrowtail \mathcal{G}'$ of labelled directed graphs $\mathcal{G}, \mathcal{G}'$ over Σ we define a homomorphism $\dot{h} : \mathcal{G} \to \mathrm{im}(h)$ by $\dot{h}(v) = h(v)$ for every $v \in V(\mathcal{G})$. This homomorphism $\dot{h}$ is an isomorphism of $\mathcal{G}$ into $\mathrm{im}(h)$, called an *isomorphism deduced by h*.

For a labelled directed graph $\mathcal{G}$ over Σ, the *identity homomorphism* (or simply, *identity of $\mathcal{G}$*), denoted by $\mathrm{id}_{\mathcal{G}}$, is the homomorphism $h : \mathcal{G} \to \mathcal{G}$ such that $h(v) = v$ for every $v \in V(\mathcal{G})$.

We say that a labelled directed graph $\mathcal{G}$ over Σ is an *isomorphically perfect* labelled directed graph over Σ if the identity homomorphism $\mathrm{id}_{\mathcal{G}}$ is a unique isomorphism of labelled directed graph $\mathcal{G}$ into $\mathcal{G}$.

Lemma 2. *Let $\mathcal{G}$ be an isomorphically perfect labelled directed graph over Σ and let $h : \mathcal{G} \to \mathcal{G}'$, $h' : \mathcal{G} \to \mathcal{G}'$ be two isomorphisms of labelled graphs $\mathcal{G}, \mathcal{G}'$ over Σ. Then $h = h'$.*

We say that a set or a class $\mathcal{A}$ of labelled directed graphs over Σ is *skeletal* if for all labelled directed graphs $\mathcal{G}, \mathcal{G}'$ in $\mathcal{A}$ if they are isomorphic, then $\mathcal{G} = \mathcal{G}'$.

A *gluing diagram $\mathcal{D}$* of labelled directed graphs over Σ is defined by:

— its *set* $\mathcal{I}$ *of indexes* with a distinguished index $\Delta \in \mathcal{I}$, called the *center* of $\mathcal{D}$,
— its *family* $\mathcal{G}_i$ $(i \in \mathcal{I})$ *of labelled directed graphs* over Σ,
— its *family* gl_i $(i \in \mathcal{I} - \{\Delta\})$ *gluing conditions* which are sets of ordered pairs such that
  (i) $\mathrm{gl}_i \subseteq V(\mathcal{G}_\Delta) \times V(\mathcal{G}_i)$ for every $i \in \mathcal{I} - \{\Delta\}$,
  (ii) $(v, v') \in \mathrm{gl}_i$ implies $\ell_{\mathcal{G}_\Delta}(v) = \ell_{\mathcal{G}_i}(v)$ for all $v \in V(\mathcal{G}_\Delta)$, $v' \in V(\mathcal{G}_i)$, and for every $i \in \mathcal{I} - \{\Delta\}$,
  (iii) for every $i \in \mathcal{I} - \{\Delta\}$ if gl_i is non-empty, then there exists a bijection

$$b_i : L(\mathrm{gl}_i) \to R(\mathrm{gl}_i)$$

  for $L(\mathrm{gl}_i) = \{v \mid (v, v') \in \mathrm{gl}_i \text{ for some } v'\}$ and $R(\mathrm{gl}_i) = \{v' \mid (v, v') \in \mathrm{gl}_i$ for some $v\}$ such that $\{(v, b_i(v)) \mid v \in L(\mathrm{gl}_i)\} = \mathrm{gl}_i$.

For a gluing diagram $\mathcal{D}$ of labelled directed graphs over Σ we define a *cocone* of $\mathcal{D}$ to be a family $h_i : \mathcal{G}_i \to \mathcal{G}$ $(i \in \mathcal{I})$ of homomorphisms of labelled directed graphs over Σ (here $\mathrm{cod}(h_i) = \mathcal{G}$ for every $i \in \mathcal{I}$) such that

$$l_{\mathcal{G}}(h_\Delta(v)) = l_{\mathcal{G}}(h_i(v'))$$

for every pair $(v, v') \in \mathrm{gl}_i$ and every $i \in \mathcal{I} - \{\Delta\}$.

A cocone $q_i : \mathcal{G}_i \to \widetilde{\mathcal{G}}$ $(i \in \mathcal{I})$ of $\mathcal{D}$ is called a *colimiting cocone of* $\mathcal{D}$ if for every cocone $h_i : \mathcal{G}_i \to \mathcal{G}$ $(i \in \mathcal{I})$ of $\mathcal{D}$ there exists a unique homomorphism $h : \widetilde{\mathcal{G}} \to \mathcal{G}$ of labelled directed graphs $\widetilde{\mathcal{G}}, \mathcal{G}$ over Σ such that $h(q_i(v)) = h_i(v)$ for every $v \in V(\mathcal{G}_i)$ and for every $i \in \mathcal{I}$. The labelled directed graph $\widetilde{\mathcal{G}}$ is called a *colimit* of $\mathcal{D}$, the homomorphisms q_i $(i \in \mathcal{I})$ are called *canonical injections* and the unique homomorphism h is called the *mediating morphism* for $h_i : \mathcal{G}_i \to \mathcal{G}$ $(i \in \mathcal{I})$.

For a gluing diagram $\mathcal{D}$ one constructs its colimit $\widetilde{\mathcal{G}}$ in the following way:

— $V(\widetilde{\mathcal{G}}) = \bigcup_{i \in \mathcal{I}} (V_i \times \{i\})$, where
  $V_\Delta = V(\mathcal{G}_\Delta)$ for the center Δ of $\mathcal{D}$,
  $V_i = V(\mathcal{G}_i) - R(\mathrm{gl}_i)$ for every $i \in \mathcal{I} - \{\Delta\}$,
— $E(\widetilde{\mathcal{G}}) = \bigcup_{i \in \mathcal{I}} E_i$, where
  $E_\Delta = \{((v, \Delta), (v', \Delta)) \mid (v, v') \in E(\mathcal{G}_\Delta)\}$ for the center Δ of $\mathcal{D}$,

$$\begin{aligned}E_i = {} & \{((v, i), (v', i)) \mid (v, v') \in E(\mathcal{G}_i) \text{ and } \{v, v'\} \subseteq V_i\} \\ & \cup \{((v, \Delta), (v', \Delta)) \mid (v, v'') \in \mathrm{gl}_i,\ (v', v''') \in \mathrm{gl}_i, \\ & \qquad \text{and } (v'', v''') \in E(\mathcal{G}_i) \text{ for some } v'', v'''\} \\ & \cup \{((v, \Delta), (v', i)) \mid v' \in V_i,\ (v, v'') \in \mathrm{gl}_i \text{ and } (v'', v') \in E(\mathcal{G}_i) \text{ for some } v''\} \\ & \cup \{((v, i), (v', \Delta)) \mid v \in V_i,\ (v, v'') \in \mathrm{gl}_i \text{ and } (v, v'') \in E(\mathcal{G}_i) \text{ for some } v''\}\end{aligned}$$

  for every $i \in \mathcal{I} - \{\Delta\}$,
— the labelling function $\ell_{\widetilde{\mathcal{G}}}$ is defined by $\ell_{\tilde{\mathcal{G}}}((v, i)) = \ell_{\mathcal{G}_i}(v)$ for every $(v, i) \in V(\widetilde{\mathcal{G}})$.

The definition of a colimiting cocone of a gluing diagram $\mathcal{D}$ provides that any other colimit of $\mathcal{D}$ is isomorphic to the colimit of $\mathcal{D}$ constructed above. Hence one proves the following lemma.

Lemma 3. *Let $\mathcal{D}$ be a gluing diagram of labelled graphs over Σ. Then for every colimiting cocone $q_i : \mathcal{G}_i \to \mathcal{G}$ $(i \in \mathcal{I})$ of $\mathcal{D}$ if $i' \neq i''$, then*

$$\big(V(\mathrm{im}(q_{i'})) - V(\mathrm{im}(q_{\Delta}))\big) \cap \big(V(\mathrm{im}(q_{i''})) - V(\mathrm{im}(q_{\Delta}))\big) = \varnothing$$

for all $i', i'' \in \mathcal{I} - \{\Delta\}$, where Δ is the center of $\mathcal{D}$ and the elements of nonempty $V(\mathrm{im}(q_i)) - V(\mathrm{im}(q_{\Delta}))$ with $i \neq \Delta$ are 'new' elements and the elements of $V(\mathrm{im}(q_{\Delta}))$ are 'old' elements.

We say that an ordered triple (k, m, n) of integers k, m, n is *acceptable* if $k > 0$, $m \neq 0$, $n > 1$, and $-k < m < n$. We define

$$\mathrm{lin}[k, n] = \big\{(i, i+1) \mid i \text{ is an integer such that } -k \leq i < -1 \text{ or } 1 \leq i \leq n\big\} \cup \{(-1, 1)\}$$

for k, n as above.

Then we say that a labelled directed graph $\mathcal{G}$ over Σ having more than one label is *induced by an acceptable ordered triple* (k, m, n) if $\mathcal{G}$ is such that

— $V(\mathcal{G}) = \big\{i \mid i \text{ is an integer such that } -k \leq i \leq n\big\}$,
— $E(\mathcal{G}) = \mathrm{lin}[k, n] \cup \{(0, m), (1, 1)\}$,
— $\ell_{\mathcal{G}}(0) \notin \{\ell_{\mathcal{G}}(k), \ell_{\mathcal{G}}(m)\}$.

For a natural number $n > 0$ a *regular labelled binary tree of depth n over* $\{\mathrm{root}, 0, 1\} \times \Sigma$ is defined to be a labelled directed graph $\mathcal{T}$ over $\{\mathrm{root}, 0, 1\} \times \Sigma$ such that

— $V(\mathcal{T})$ is the set of binary strings[6] of length not greater than n including empty string Λ,
— $E(\mathcal{T}) = \big\{(\Gamma, \Gamma i) \mid \{\Gamma, \Gamma i\} \subseteq V(\mathcal{T}) \text{ and } i \in \{0, 1\}\big\} \cup \big\{(\Gamma, \Gamma) \mid \Gamma \text{ is a binary string of length } n\big\}$,
— the labelling function $\ell_{\mathcal{T}} : V(\mathcal{T}) \to \{\mathrm{root}, 0, 1, \bot\} \times \Sigma$ of $\mathcal{T}$ is such that $\ell^1_{\mathcal{T}}(\Lambda) = \mathrm{root}$, $\ell^1_{\mathcal{T}}(\Gamma i) = i$ for every binary string Γ and every $i \in \{0, 1\}$ such that $\Gamma i \in V(\mathcal{T})$,

where $\ell^1_{\mathcal{T}}(x)$, $\ell^2_{\mathcal{T}}(x)$ denote the coordinates such that $\ell_{\mathcal{T}}(x) = (\ell^1_{\mathcal{T}}(x), \ell^2_{\mathcal{T}}(x))$ and Γi denotes that binary string Θ whose last element is the digit i, and Γ is that binary string which is the result of deleting the last element in Θ.

Lemma 4. *The set of labelled directed graphs over Σ induced by acceptable ordered triples of integers is a skeletal set of isomorphically perfect graphs for Σ having more than one label.*

[6] A binary string is a sequence, maybe empty, of digits 0, 1.

Lemma 5. *The set of all regular binary trees of arbitrary depth over* $\{\mathrm{root}, 0, 1\} \times \Sigma$ *is a skeletal set of isomorphically perfect graphs.*

We adopt the following notation for a set X of ordered pairs of natural numbers:

$$\begin{aligned} i \odot X &= \{(m \cdot 2^i, n+i) \mid (m,n) \in X\} \text{ for a natural number } i,\\ (0,1) \oplus X &= \{(m, n+1) \mid (m,n) \in X\},\\ (1,1) \oplus X &= \{(m+2^n, n+1) \mid (m,n) \in X\}, \end{aligned}$$

we recall that subtraction $\dot{-}$ of natural numbers is given by

$$m \dot{-} n = \begin{cases} 0 & \text{if } n > m,\\ m-n & \text{if } n \geq m. \end{cases}$$

We define inductively the finite sets T_n^m of ordered pairs of natural numbers for natural numbers m, n by the following equations:

(t_0) $T_n^0 = \{(0,i) \mid i \text{ is a natural number with } 0 \leq i \leq n\}$,
(t_1) $T_0^m = T_0^0$,
(t_2) $T_n^{n-i} = T_i^0 \cup (i \odot T_{n-i}^{n-i})$ for $0 < i < n$,
(t_3) $T_n^{n+i} = T_0^0 \cup \big((0,1) \oplus T_{n-1}^{(n \dot{-} 3)+i}\big) \cup \big((1,1) \oplus T_{n-1}^{(n \dot{-} 3)+i}\big)$

for $n > 0$ and $i \geq 0$.

Lemma 6. *For all natural numbers* $n \geq 2$ *and* $i > 0$

$$T_n^{3 \cdot n - 3 + i} = T_n^{3 \cdot n - 3}.$$

Proof. We prove the lemma by induction on n. □

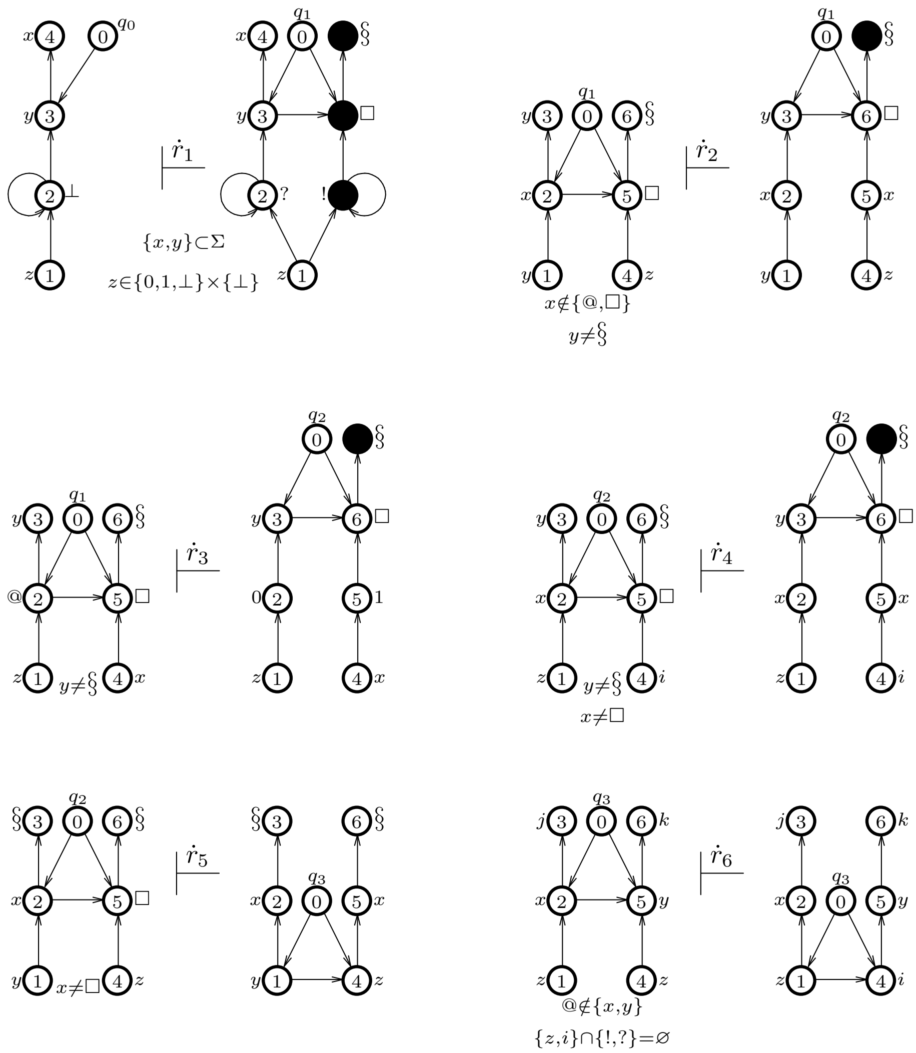
For natural numbers m, n we define labelled directed graph $\mathcal{T}_n^m$ over $\{\bot, \mathrm{root}, 0, 1\} \times \{0, 1, \bot\}$ by

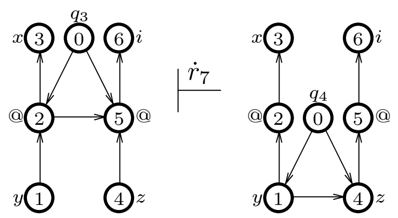
— $V(\mathcal{T}_n^m) = T_n^m$,
— $E(\mathcal{T}_n^m) = E_1^{m,n} \cup E_2^{m,n}$ for
$E_1^{m,n} = \big\{\big((i,j),(i',j')\big) \mid \{(i,j),(i',j')\} \subset T_n^m,\ j' = j+1,$
$i \leq i'/2$, and $i = \max\{k \mid (k,j) \in T_n^m,\ \text{and } k \leq i'/2\}\big\}$,
$E_2^{m,n} = \big\{\big((i,n),(i,n)\big) \mid (i,n) \in T_n^m\big\}$,
— the labelling function $\ell_{\mathcal{T}_n^m}$ of $\mathcal{T}_n^m$ is such that
$\ell_{\mathcal{T}_n^m}((i,j)) = \big(\ell^1_{\mathcal{T}_n^m}((i,j)), \bot\big)$ for $\ell^1_{\mathcal{T}_n^m}((i,j))$ defined by
(t'_0) $\ell^1_{\mathcal{T}_n^m}((0,0)) = \mathrm{root}$,
(t'_1) if $x \in T_n^m$ with $x \neq (0,0)$ and $P_x = \big\{y \mid \{(z,y),(z,x)\} \subset E_1^{m,n}$ for some $z\big\}$ has exactly one element, then $\ell_{\mathcal{T}_n^m}(x) = \bot$,
(t'_2) if $x \in T_n^m$ and $P_x = \{(i,j),(i',j)\}$ with $i < i'$, then $\ell^1_{\mathcal{T}_n^m}((i,j)) = 0$ and $\ell^1_{\mathcal{T}_n^m}((i',j)) = 1$.

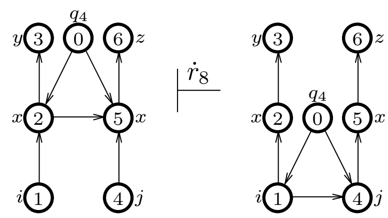
Lemma 7. *For every natural number $n \geq 2$ the labelled directed graph $\mathcal{T}_n^{3\cdot n-3}$ is isomorphic to a regular labelled binary tree $\mathcal{T}$ of depth n over $\{\bot, \mathrm{root}, 0, 1\} \times \{0, 1, \bot\}$ such that $\ell^2_{\mathcal{T}}(\Gamma) = \bot$ for all $\Gamma \in V(\mathcal{T})$.*

Proof. We prove the lemma by induction on $n \geq 2$. ☐

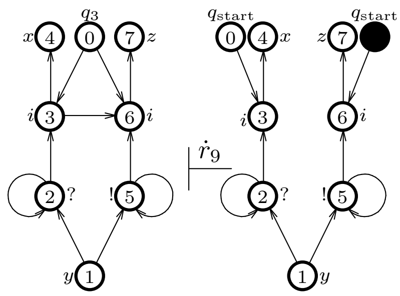
The schemes $(\dot{r}_1)$–$(\dot{r}_{12})$ of the rewriting rules are illustrated by the figures below, where the numbers in circles are the vertices of the corresponding labelled graphs, the new vertices are indicated by bold dots •, the labels of the vertices stand close to the corresponding circles, and ?, !, $\bot$ are the abbreviations of the labels $(0, \bot)$, $(1, \bot)$, $(\bot, \bot)$, respectively. The variables x, y, z, i, j, k range over the set $\Sigma^\bullet$ of labels.



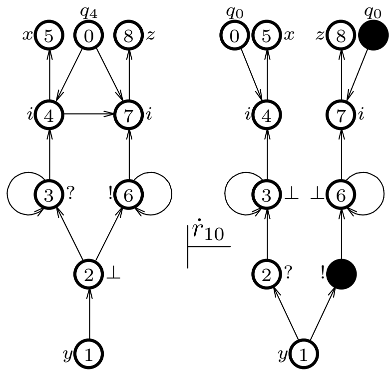




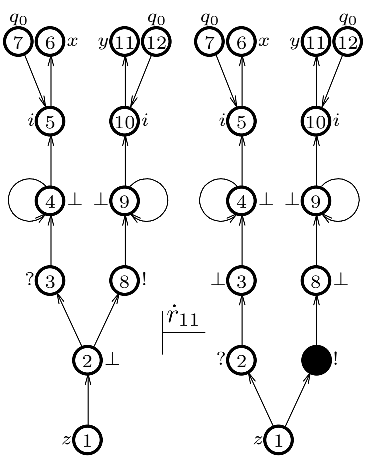
$\{i,j\}\cap\{?,!\}=\varnothing$



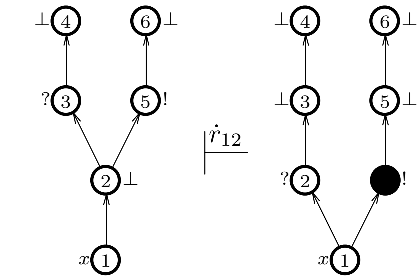
$y\in\{0,1\}\times\{\bot\}$



$y\in\{0,1,\bot\}\times\{\bot\}$



$z\in\{0,1,\bot\}\times\{\bot\}$



$x\in\{0,1,\bot,\text{root}\}\times\{\bot\}$

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