An algorithmic approach to tilings of hyperbolic spaces: 10 years later

(Abstract)

Maurice MARGENSTERN

Université Paul Verlaine – Metz, IUT de Metz,
LITA EA 3097, UFR MIM,
Campus du Saulcy,
57045 METZ Cédex 1, FRANCE
e-mail: margens@univ-metz.fr
Web page: http://www.lita.sciences.univ-metz.fr/~margens

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Ten years ago, a paper appeared in the *Journal of Universal Computer Science* which laid the basis of an algorithmic approach to tilings of the hyperbolic plane. This approach appeared to be very fruitful and it gave rise to around 85 papers in journals, conferences and workshops. A few other papers also appeared connected with other approaches to such tilings, motivated by a combinatorial point of view too.

In the talk, we give an account of the works of the algorithmic approach developed by the author and to possible applications.

In a first part, the talk will remind what is needed to know from hyperbolic geometry in order to understand the results which will be indicated. The reader will have to remind the frame of Poincaré's disc model only. Here, we simply refer to Figure 1 and its caption.

In a second part, we shall explain the method on the classical case of the pentagrid, the tiling $\{5,4\}$ of the hyperbolic plane which is obtained from the regular rectangular pentagon by reflection in its sides and, recursively, by reflection of the images in their sides. We shall see that the tiling is spanned by a tree and that numbering the nodes of the tree in an appropriate way together with a suited representation of the numbers will make an important property appear from which we can easily derive **navigation tools** in the tiling. Below, Figure 2

A remarkable property of the hyperbolic plane is that the navigation tools obtained for the pentagrid work exactly in the same way in the heptagrid, the tiling $\{7,3\}$ of the hyperblic plane, and, with a simple adaptation, to all tilings of the hyperbolic plane of the form $\{p,4\}$ and $\{p+2,3\}$. The method also allows to study all tilings $\{p,q\}$ of the hyperbolic plane. It was also extended to the hyperbolic $3D$ and $4D$ spaces, allowing us to obtain a very valuable information.

In a third part, the talk will indicate the results obtained in terms of tilings. The main results are the representation of the 120-cell grid: the tiling $\{5,3,3,4\}$ of the hyperbolic $4D$ space; the undecidability of the tiling problem of the hyperbolic plane and connected results.

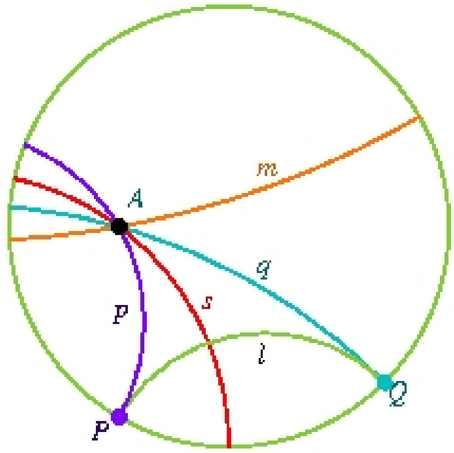

Figure 1 *Poincaré's disc model. The points of the hyperbolic plane are the points of the open unit disc U whose border, ∂U, consists of the* **points at infinity** *and is represented by the green circle. Lines are trace on the open disc of diameters or circles which are orthogonal to ∂U. They are parallel if and only if they have a common point at infinity. In the figure, we can see two parallels to ℓ which pass through A. Also note the line m which also passes through A but which does not cut ℓ, neither in U, nor on ∂U, nor outside U.*

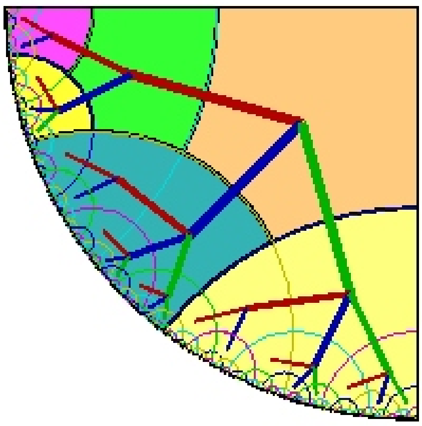

Figure 2 *The underlying tree of the tiling: the nodes of the tree are in bijection with the tiles which exactly cover a quarter of the plane.*

In a fourth part, the talk will consider the application of the navigation tools to the implementation of cellular automata in hyperbolic tilings. The obtained results can be divided into three categories: general properties, complexity results and universality results. We shall briefly mention the universality results which were investigated for the pentagrid, the heptagrid and the dodecagrid. This latter tiling is the tessellation of the hyperbolic $3D$ space obtained from what is called Poincaré's dodecahedron whose faces are copies of the regular rectangular pentagon. Figure 3 represents the basic bricks which allow us to construct this tilings. All universality results which will be presented will be based on the implementation of a railway circuit in the considered hyperbolic grid.

Figure 3 *The lego-like bricks for the dodecagrid.*

In a fifth and last part, we shall investigate possible applications of the navigation tools. In a first sub-part, we shall look at already obtained applications. Then, in the second sub-part, we shall look at possible applications to biology. One application is theoretical and deals with an implementation of P systems, it was already presented a few years ago at the occasion of an edition of this conference. The second application is more practical. It is just a proposal which would require a rather huge effort.